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Potassium Bioxalate and the Standardisation of Alkali Solutions.

By

Yukichi Osaka and Kinji Ando.

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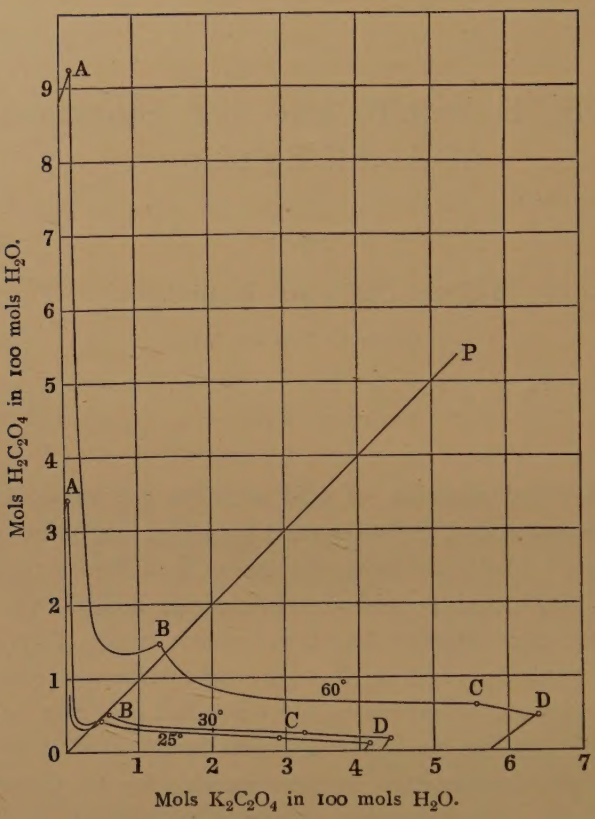
For the standardisation of acid solutions anhydrous sodium carbonate is recommended in most text-books of analytical chemistry, but for that of alkali solutions the matter is not so simple. Oxalic acid, $H_2C_2O_4.2H_2O$, and potassium tetraoxalate, $KH_3(C_2O_4)_2.2H_2O$, are often used for that purpose, but they contain water of crystallisation and it is not always certain that we have them perfectly dry and with no loss of water of crystallisation. Succinic acid may be obtaineed in a pure state and has no water of crystallisation and it is generally used among physico-chemists as recommended by W. Ostwald in his *Physiko-chemische Messungen*, but it is not in general use. Thus for the standardisation of alkali solutions an acid or acid salt in a solid state, which crystallises without water of crystallisation and can be obtained pure in every laboratory without much difficulty, is much to be desired.

Potassium bioxalate crystallises anhydrous at a temperature above 15°, and if it could be prepared easily in pure state, it might be used conveniently for the standardisation of alkali solutions. Its solutions are, however, incongruently saturated at 25° and 30° and it can not be purified at those temperatures by simple crystallisation. But it forms a congruently saturated solution at 60°, as may be seen from the accompanying diagram. For this diagram the data given in the literature were re-calculated for our purpose.[1] The number of mols of oxalic acid and of potassium oxalate in 100 mols of water is measured

1 H. W. Foote snd J. A. Andrew, Amer. Chem. J., **34**, 153 (1905); J. Koppel and M. Cahn, Zs. anorg. Chem., **60**, 54 (1908).



along the axes of the ordinate and abscissa respectively. The curves, AB, BC, and CD, represent respectively the solubilities of the acid salts, $KH_3(C_2O_4)_2.2H_2O$, KHC_2O_4, and $K_4H_2(C_2O_4)_3.2H_2O$, in presence of an excess of either component. The line, OP, bisects the angle between the axes and any pure solution of potassium bioxalate may be represented by a point on the line. From the diagram it may be estimated that at a temperature not very far from 40° the bioxalate begins to form congruently saturated solutions.

In the solutions having the bioxalate as the residue the ratio of oxalic acid and the potassium salt varies between 1 : 1·2 and 1 : 13 at 25° and 1 : 0·9 and 1 : 9 at 60°, so we have in common between these two temperatures the ratio between 1 : 1.2 and 1 : 9. Thus if we take oxalic acid, potassium oxalate, and water in proportions to form a saturated solution of the bioxalate at a certain temperature between 25° and 60° containing the acid and salt in a ratio within the above

stated limits, then any amount of them, added in equivalent quantities and in an anhydrous state to the solution, should separate out as the bioxalate.

Guided by the above principle, the preparation of the bioxalate was tried. As the crystallisation was made to take place at 25°, a solution containing 0·4 mol of oxalic acid and 0·6 mol of the potassium salt in 100 mols of water was chosen as the medium of the crystallisation, and one, two, or five mols of the bioxalate were made to crystallise from it by adding the corresponding quantities of the components in equivalent ratio. The proportions of the materials to be used and the amounts actually taken are given in the following table.

TABLE I.

	Proportions to be used. Mols.	Proportions to be used. Grams.	Actually taken. Grams.
	I.		
H_2O	95·6	1721	500
$K_2C_2O_4.H_2O$	1·6	294·7	80
$H_2C_2O_4.2H_2O$	1·4	176·4	51
	II.		
H_2O	92·6	1666	300
$K_2C_2O_4.H_2O$	2·6	478·9	86
$H_2C_2O_4.2H_2O$	2·4	302·4	54·5
	III.		
H_2O	83·6	1505	150
$K_2C_2O_4.H_2O$	5·6	1031	103
$H_2C_2O_4.2H_2O$	5·4	680·5	68

The experiments were carried out in the following way:

I. Oxalic acid and potassium oxalate were each thrice recrystallised. The calculated quantities of the materials were taken in an Erlenmeyer flask of about 1000 cc. capacity, stoppered with a cork with a long capillary tube and heated in a water bath to complete dissolution. The stopper was replaced by an ordinary cork and kept in a thermostat at 25° for two days. Then the greater part of the mother liquor was poured off by decantation, the crystals were transferred to filter paper placed on the perforated plate of a porcelain

funnel and the adhering mother liquor was removed by means of a water suction pump. The crystals were washd three times with 50 cc. of water at about 50°, pressed between folded filter paper and finally dried in an air bath at about 110° to a constant weight. The yield was 60 grams against the calculated value of 74, namely 82 per cent.

II, An Erlenmeyer flask of about 600 cc. capacity was used, and as the acid and salt were not completely dissolved the mixture was heated for about one hour. On other points the operation was just the same as in the previous case. The yield was 75 grams against the calculated value of 92, namely 82 per cent.

III. An Erlenmeyer flask of abont 500 cc. capacity was used. As thc greater part of the acid and salt remained undissolved, the mixture was heated for about two hours. For washing of the crystals 200 cc. of water at about 50° were used in three portions. The yield was 107 grams against the calculated value of 128, namely 84 per cent.

The yields in all three cases amounted to about 83 per cent. of the calculated value, and this would be unavoidable as the washing of the crystals requires hot water.

To test for the purity of the bioxalate obtained its decinormal solution as to the acidic hydrogen was prepared and compared with a decinormal solution of succinic acid against a dilute baryta water with phenolphthalein as indicator. The bioxalate solution and the succinic acid solution proved to be exactly in the same equivalent concentration.

The percentage amount of the potassium in the bioxalate obtained was then determined by converting it into the sulphate in the usual way and the following values were obtained against the theoretical value of 30·52 per cent.

I. 30·43, 30·42, 30·42, and 30·43.
II. 30·43, 30·42, 30·45, and 30·45.
III. 30·42, 30·45, 30·45, and 30·43.

As may be seen from the two kinds of testing, there is no difference among the three samples. The values found for the potassium are somewhat low, but as this difference corresponded to a loss of about one milligram in the sulphate and the conversion of the oxalate into

the sulphate is very liable to loss by splitting, the samples may be taken as pure sufficiently at least for the purpose of the standardisation of alkali solutions.

If no thermostat is available, the following procedure may be taken among others. As roughly estimated from the diagram we have a saturated solution of the bioxalate which contain 1·1 mol of the acid and 1·5 mol of the potassium salt in 100 mols of water at 60° and this system is still in equilibrium with the bioxalate as the residue when the temperature is lowered to any degree between 60° and 25°, the excess of the acid and salt separating out as the bioxalate. Thus if we take 80 grams of oxalic acid ($H_2C_2O_4 \cdot 2H_2O$), 160 grams of potassium oxalate ($K_2C_3O_4 \cdot H_2O$), any amount of the mixture in equivalent ratio, and 1000 grams of water diminished by its amount in the equivalent mixture as the water of crystallisation, and after treating the mixture as in the experiments above described, separate the crystals from the mother liquor at any temperature between 60° and 25°, we can obtain the bioxalate in pure state. This lower limit of the temperature can probably be lowered still more, but never lower than 15°.

Summary.

From the standpoint of the phase rule a method of the preparation of pure anhydrous potassium bioxalate was devised and that acid salt is recommended for the standardisation of alkali solutions.

Equilibrium in the System: Potassium Chlorate, Potassium Nitrate and Water at 25°C.

By

Shigeru Toda.

(Received October 19, 1920).

According to Herbette,[1] two kinds of crystals separate when a mixed solution of potassium chlorate and nitrate is evaporated. The one is monoclinic and contains both salts, and the other is rhombic and consists of the pure nitrate. Thus, whilst potassium nitrate may take an unstable form isomorphous with the ordinary form of the chlorate, there seems to be no second form of the latter isomorphous with the stable form of the former.

From the above cited fact, a systematic investigation of the system consisting of potassium chlorate, potassium nitrate, and water was desirable to add an example of a special type to the literature of the heterogeneous equilibrium of salts and water, the author took up the problem to study it from the standpoint of the phase rule.

As the materials, potassium chlorate and nitrate of the Jap. Pharm. purified by thrice recrystallisation, and ordinary distilled water were used.

The experiments were carried out in the following way: Mixtures of salts in different proportions were put each in an Erlenmeyer flask of a capacity of about 30 cc. with a well ground stopper, which was made to rotate in a thermostat at 25·0°C, for about two days and nights. When equilibrium was attained, the flask was allowed to

1 M. J. Herbette, C.R., 143. 128–130, (1906).

stand still in the same thermostat until the suspended matter had completely subsided, then a proper portion of the mother liquor was taken out by a pipette, through a short glass tube with purified cotton wool and attached to the pipette by a rubber tubing, to a weighing bottle and subjected to analysis.

To prepare the residue for analysis, it was pressed down between folded filter paper to remove as much as possible of the adhering mother liquor, then a part of it, previously weighed, was redissolved in water to a definite volume.

The methods of analysis used were as follows: Exactly 10 cc. of the solution, in a beaker, was diluted with about 30 cc. of water, treated with 40 cc. of a 10 per cent. solution of ferrous sulphate, heated with constant stirring till it began to boil, and kept at that temperature for about fifteen minutes. Aqueous ammonia which was absolutely free from chlorine was added into this solution to remove ferric salt. It was gently boiled until the excess of ammonia was expelled, then the precipitate was filtered and washed five times with hot water. The filtrate was used for the determination of the chlorine according to Volhard's method, improved by V. Rothmund and A. Burgstaller.[1] Potassium was determined as potassium sulphate by means of sulphuric acid, as directed in Treadwell-Hall's "Analytical Chemistry"[2]

Results.

The results are given in the following table:—

TABLE I.

	Composition of solution in gram percentages.		Composition of residue[3] in gram percentages.	
No.	$KClO_3$	KNO_3	$KClO_3$	KNO_3
1	7·745	...	...	...
2	7·65	0·68	94·44	0·81
3	7·07	1·55	93·95	1·90
4	6·52	3·59	93·37	2·83
5	5·76	7·12	93·65	4·85
6	5·10	12·81	91·44	6·80

[1] V. Rothmund and A. Burgstaller: Z. anorg. Chem., **63**. 330, (1909).

[2] Treadwell-Hall: Analytical Chemistry. Vol. II, 41.

[3] The residue was slightly wet with the mother liquor.

7	4·39	18·97	88·85	9·11
8	3·90	27·14	86·05	11·90
9	3·90	27·12	50·65	43·63
10	3·90	27·14	19·13	76·96
11	3·90	27·16	9·54	86·96
12	3·61	27·21	0·17	95·98
13	1·63	27·57	0·33	95·18
14	...	27·24	...	...

The following table was calculated from the last, using the following formula to represent the compositions of the solutions and residues:

$$100m\mathrm{H_2O}\ x\mathrm{KClO_3}\ (100\text{-}x)\mathrm{KNO_3}$$

and in the table, a function of m, $N=\frac{100m}{100+m}$, was calculated to represent the amount of water.

TABLE II.

	Solution.		Residue.		
No.	N	x	N	x	Formula of the residue found graphically.
1	44·76	100·00	...	...	$KClO_3$
2	42·25	90·29	0·34	98·97	$K(ClO_3, NO_3)$
3	41·00	79·01	0·29	97·62	,,
4	36·00	60·00	0·26	86·47	,,
5	29·17	40·01	0·10	94·11	,,
6	21·30	24·73	0·12	91·73	,,
7	16·00	16·03	0·14	88·95	,,
8	11·30	10·59	0·14	85·63	,,
9	11·32	10·59	0·37	48·92	$K(ClO_3, NO_3)+KNO_3$
10	11·30	10·59	0·24	17·01	,, ,,
11	11·53	10·84	0·21	8·30	,, ,,
12	11·39	9·85	0·22	0·14	KNO_3
13	12·08	4·64	0·26	0·29	,,
14	13·03	0·00	...	...	,,

The above results are graphically shown in Fig. 1, in a square diagram, measuring water (N) on the vertical side and molar percentages of the chlorate (x) on the horizontal side.

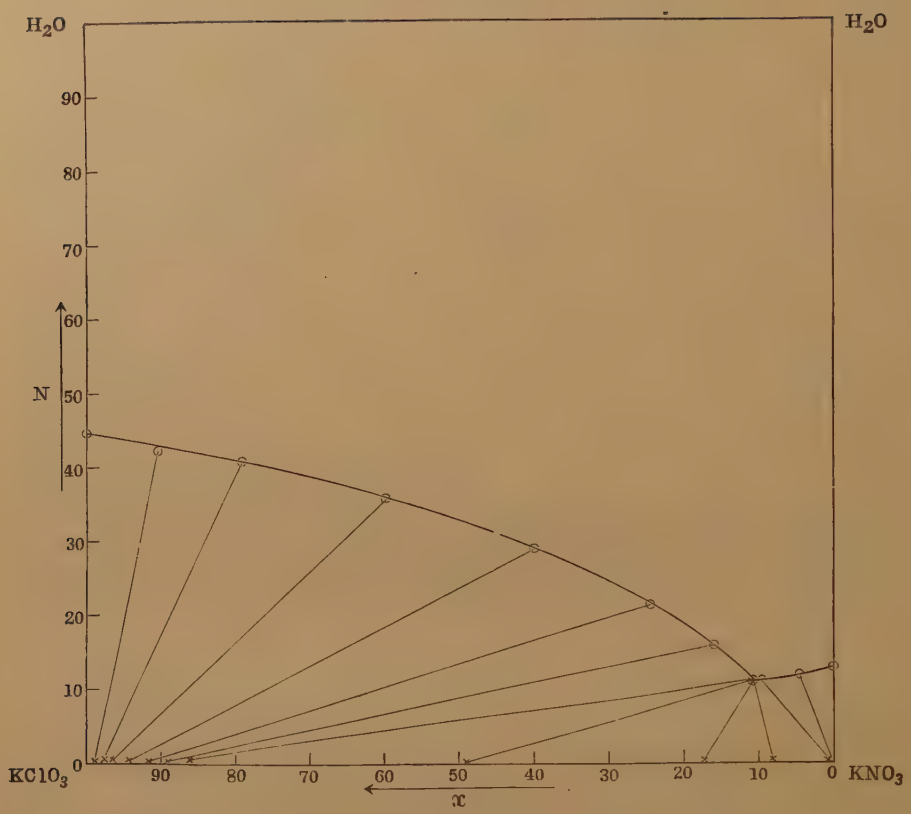


Fig. 1.

The following table was calculated from Table I, using the following formulae to represent the molar percentages of the salts in the solutions and residues:

$x KNO_3\ (100-x)KClO_3$ Solution,
$y KNO_3\ (100-y)KClO_3$ Residue.

Table III.

	Solution.	Residue.
No.	x	y
1	0·00	0·00
2	9·71	1·03
3	20·99	2·38

4	40·00	3·53
5	59·99	5·89
6	75·27	8·27
7	83·97	11·05
8	89·41	14·37
9	89·41	51·08
10	89·41	82·99
11	89·16	91·70
12	90·15	99·86
13	95·36	99·71
14	100·00	100·00

The above results are graphically shown in Fig. 2, in a square diagram, measuring molar percentages of the nitrate in the solutions (x) on the vertical side and those in the residues (y) on the horizontal side.

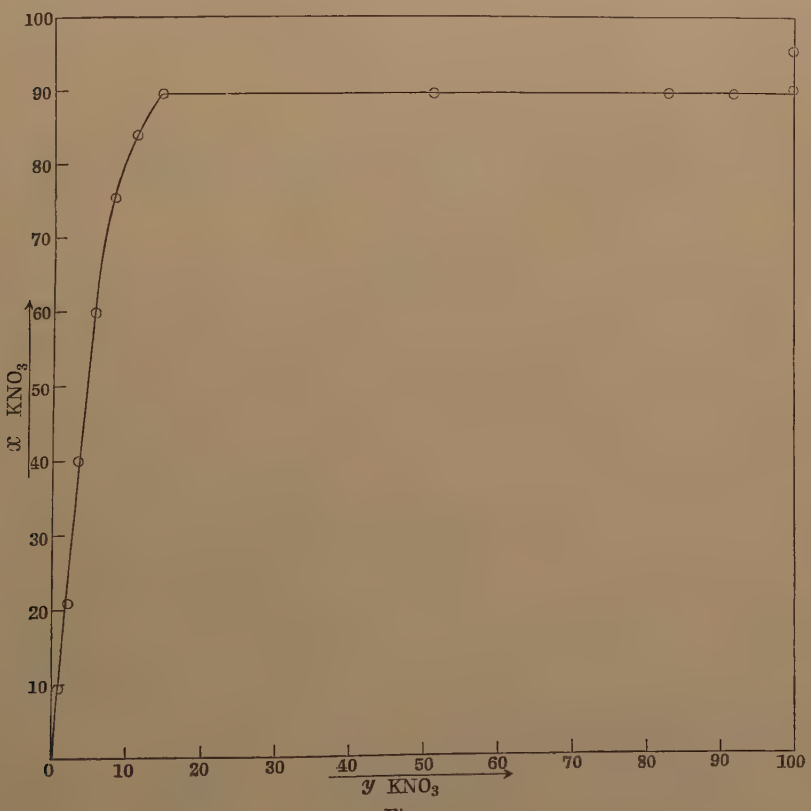


Fig. 2.

The crystalline form of the solid solution is monoclinic. From this fact it may be presumed that this mixed crystal is of the same kind as the crystal which was obtained by Herbette[1] by evaporation of a solution containing 15 grams of chlorate and 100 grams of nitrate. As the monoclinic crystals, separated by Herbette, contained 65 per cent. of the chlorate, the molar percentage of the nitrate in them is calculated as 39·47, but the mixed crystals obtained in the present investigation contains as the maximum only 14·37 of potassium nitrate in molar percentage (Fig. 2). This difference may be principally due to the difference of temperature.

Summary.

The equilibrium of a system consisting of potassium chlorate, potassium nitrate, and water was studied at 25·0°C.

The solubility of potassium chlorate was determined to be 7·745 grams per 100 grams solution or 8·395 grams in 100 grams water, and that of the potassium nitrate to be 27·24 grams per 100 grams solution or 37·44 grams in 100 grams water.

At 25°C. potassium chlorate takes up potassium nitrate to form a solid solution to an extent of 14·37 in molar percentage, while the nitrate takes up none of the chlorate.

This experiment was suggested by Prof. Y. Osaka and carried out under his direction. The author wishes to take this opportunity to thank him for his helpful suggestions and kind instructions.

1 *Loc. cit.*, 128–130.

Luminosity produced by Electric Discharges in Tubes containing Gases at slightly reduced Pressures

By

Gisaburo Nakamura

(Received Nov. 11, 1920)

When an electric discharge takes place in a tube containing a gas at a reduced pressure, the positive column shows striation. If the pressure of the gas be increased the striae get closer and closer, and then a thin and apparently unstriated glow appears along the axial portion of the tube. On further increase of the pressure this glow is then branched into a number of finer ones, and the discharge assumes a noisy character. When the glows were analysed by a rotating mirror, the striation of the positive column appeared to remain almost stationary, but the thin unstriated one occurring at a pressure of a few centimeters got curved, showing that a luminosity has apparently started at the anode and then proceeded toward the cathode. The nature of this luminosity was studied and the result obtained so far is described below.

A tube provided with aluminium wire electrodes was prepared from glass tubing having a diameter about 1·5 cm. This was connected to a pump and a manometer, and excited by a Blitzen transformer of $\frac{1}{4}$ K.W., a glow oscilloscope and a milliammeter being inserted in the circuit. A visual study of the process of the discharge was made by means of a rotating mirror and its successive stages were photographed

with a falling plate camera. In some cases, vibrations of a tuning fork were recorded simultaneously on the photographic plates to determine the velocity of the latter. The arrangement of the discharge tube and the optical system is shown in the accompanying figure.

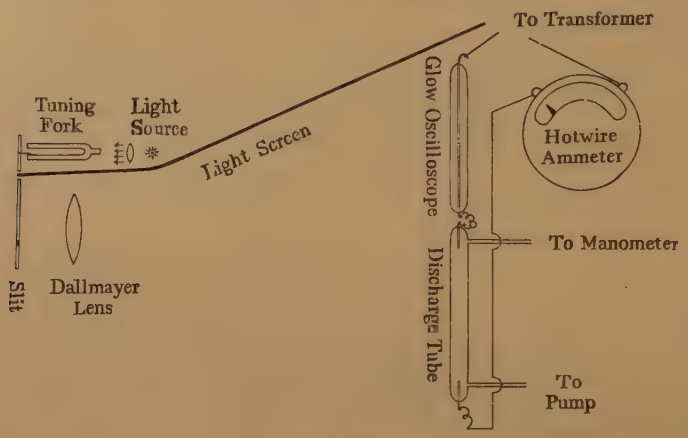


Fig. 1.

In a tube containing the air, the positive column was uniform at the pressure of 1·5 mm., the distance between the electrodes being about 5 cm. A photograph taken with the falling plate camera shows that the column really consists of uniformly illuminated bands, and consequently we may consider the light as almost simultaneously emitted from every point of the positive column. But when the pressure was increased to 2·0 mm. a slight horizontal concentration of the light was observed on the plate at the initial stage of the discharge. This got brighter and began to curve as the pressure was further increased, and then detached

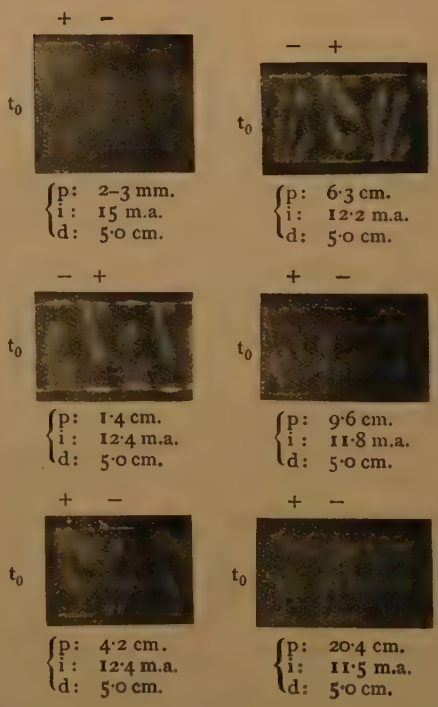


Fig. 2.

from the remainder, when the pressure was raised to 1 cm. This will be seen from the photographs shown in Fig. 2. In all these and the following figures, the sides marked by t_0 represent the lower edges of the falling plates; "+" and "−" anode and cathode of the tube respectively, "p" denoting the pressure in the tube, "i" the current and "d" the electrode distance. Thus the luminosity seems to start at the anode and proceed toward the cathode at the beginning of the discharge, the apparent velocity being about 70 m/sec. at its start. Further increasing the pressure, and keeping the current constant, the second luminous band appeared at the pressure of 1·4 cm., and the third one at a still higher pressure. The number of these luminous bands and their thickness increased with the pressure, and when the current was raised, keeping the pressure constant, their number also increased and they approached more to the cathode, but did not reach it.

The following table shows how the number of the bands and their lengths vary with the currents when the pressure was kept constant at 4·2 cm., the distance of the electrodes being 11·5 cm.

Currents	Number of Bands	Lengths of Bands				
		1st	2nd	3rd	4th	5th
4·0 m.a.	2	7·7 cm.	1·9			
5·2	3	10·2	3·8	0·6		
6·0	4	10·9	7·0	3·8	0·9	
8·0	5	11·2	9·6	6·4	3·5	0·9

The distribution of light in one of these luminous bands was not uniform; its convex side was generally brighter, and the light gradually faded away toward the concave side of the band. On the whole, the cathode side of the band was brighter, and lined by a thin but bright layer. Besides this, it was a cathode that is the brightest part of the band.

The formation of these luminous bands is not only not restricted to the air, but also in coal gas such bands appeared in a remarkable way. Photographs shown in Fig. 3 represent successive stages of the discharge in the coal gas when pressure was varied from 2·0 mm. to 82.0 mm. In a tube containing hydrogen only one such band was observed at a pressure of about 7 cm., and no more, while in oxygen no such luminous band could be detected.

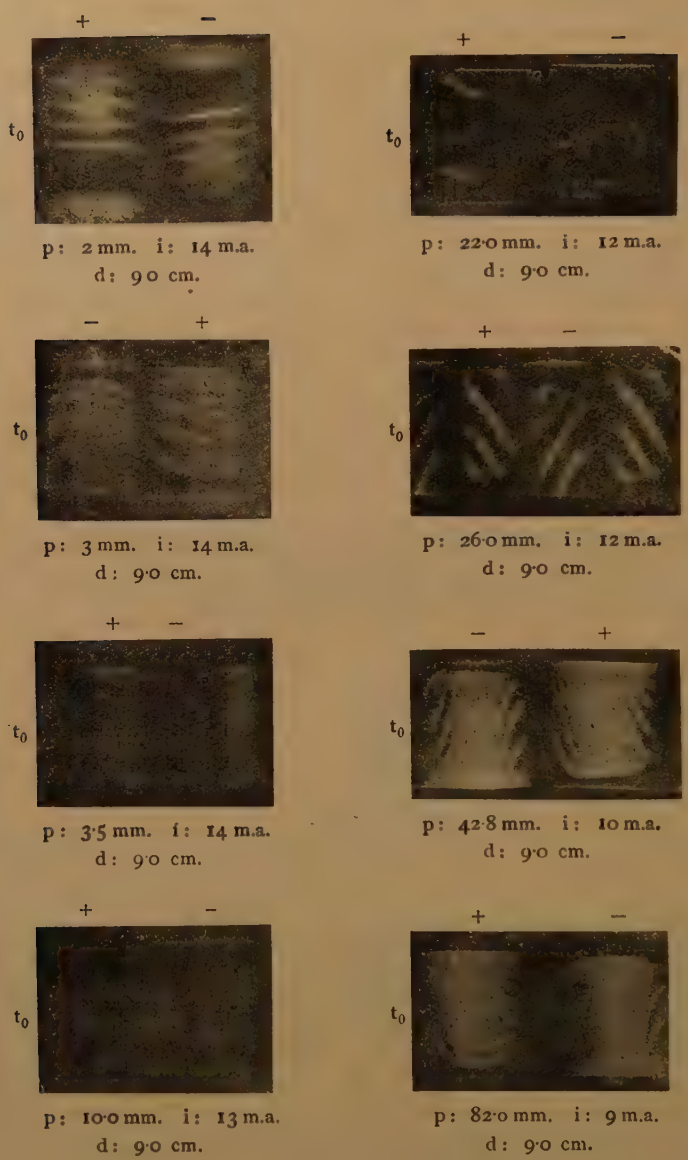

+
−
t_0
p: 2 mm. i: 14 m.a.
d: 9 0 cm.
p: 22·0 mm. i: 12 m.a.
d: 9·0 cm.
p: 3 mm. i: 14 m.a.
d: 9·0 cm.
p: 26·0 mm. i: 12 m.a.
d: 9·0 cm.
p: 3·5 mm. i: 14 m.a.
d: 9·0 cm.
p: 42·8 mm. i: 10 m.a.
d: 9·0 cm.
p: 10·0 mm. i: 13 m.a.
d: 9·0 cm.
p: 82·0 mm. i: 9 m.a.
d: 9·0 cm.

Fig. 3.

Now, it appears from the photographs given above that the luminosity was set up in the neighborhood of the anode, and thence proceeded toward the cathode. A question then naturally arises: how does it so proceed? At first sight, this seems to resemble the steamer

observed by Schuster and Hemsalech[1] in a spark discharge. According to Milner[2] it is the cathode from which the steamer emanates, but he also remarked that there is, however, in all cases a tendency to a discharge from the anode, and that with soft metals this becomes very pronounced, being very strong, as in the case of the cathode steamer. In our case, it is, however, the anode from which the luminosity apparently started, and no luminosity starting from the cathode could be detected. Thus these luminosities seem not to be related to the so-called steamer of Schuster and Hemsalech.

Next, the supposed velocities with which the luminosity proceeded from the anode were roughly measured from their inclinations, the velo-

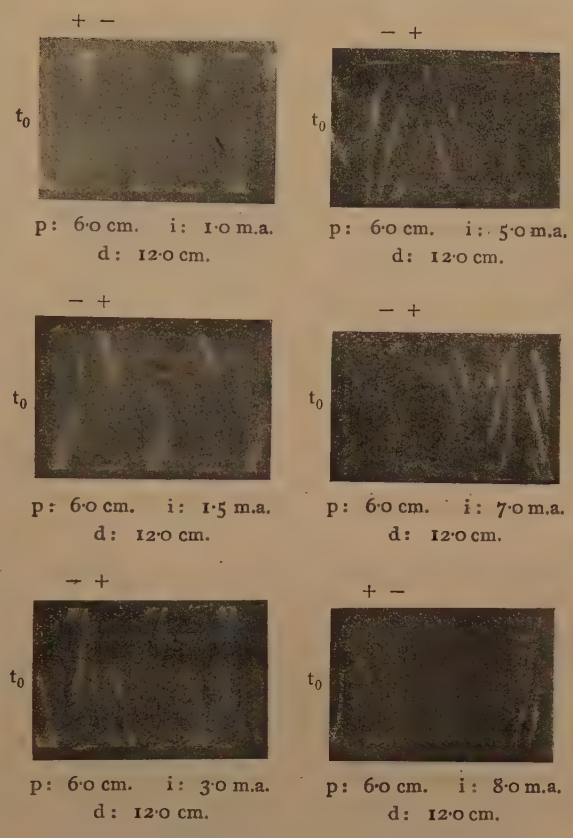


Fig. 4, a.

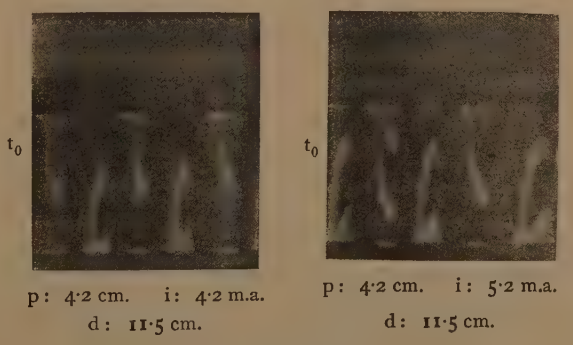
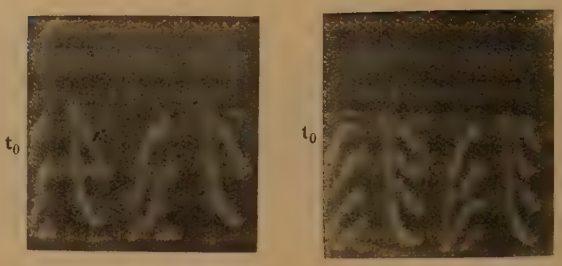


Fig. 4, b.

1 Phil. Trans., **193** A, 189 (1900).

2 Phil. Trans., **209**, 71 (1909).

city of falling plates being determined by means of tuning fork recorded on the photographic plates as shown in Fig. 4, b. In a certain case, the initial velocities of the first five luminosities were found to be 75·0, 66·3, 55·3, 40·9, and 37·5 m/sec. respectively, the pressure being about 20 cm. These values appear to be too small for velocities of positive atomions acquired by passing through the anode fall, and also it is hardly conceivable that luminous particles should have travelled from the anode to such great distances in the air at a pressure of 20 cm.



p: 4·2 cm. i: 6·8 m.a. d: 11·5 cm.

p: 4·2 cm. i: 8·0 m.a. d: 11·5 cm.

Fig. 4, b.

The maximum distance travelled by the first luminosity increased with the pressure when the current was kept constant. This is clear from the preceding figures. This maximum distance also increased with the current in the tube when the pressure was kept constant as shown in Fig. 4. The accompanying curves also represent these two relations.

The first of these facts is inconsistent with the idea that the luminosity is due primarily to the anode. The phenomenon can be explained by assuming that the luminosity is caused by ionization due to electrons coming from the

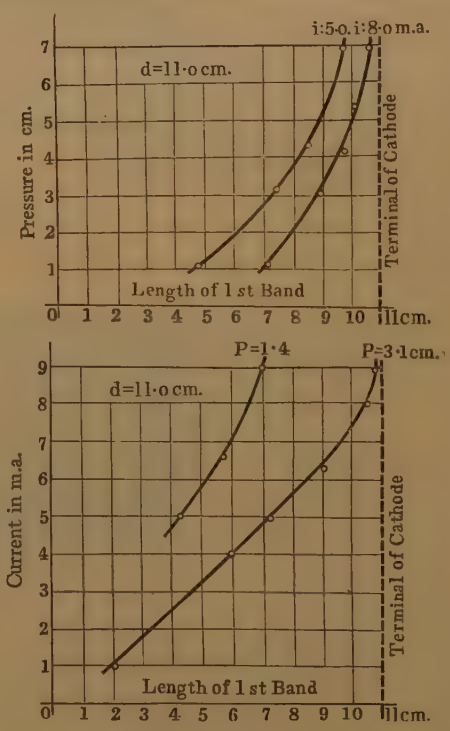


Fig. 5.

cathode. Beside this, all the facts mentioned before, namely, that the luminous bands got broader as the pressure was increased, that cathode ends were thicker than the remainder, and that the cathode side of the band was brighter and lined by a thin but bright layer, are all in harmony with the above mode of explanation.

This is confirmed by the plates obtained when the tube was excited by an induction coil. Reproductions of such photographs were shown in Fig. 6. It will be noticed that the discharge consisted of a series of partial ones, and that the first discharge made its whole path uniformly luminous, while the following ones gave alternately bright and dark spots. Thus the discharge, when analysed by a rotating mirror, presents an appearance exactly as if the luminosity proceeded toward the cathode. But the real nature of the phenomenon is that a luminous place changed its position at successive instants.

The difference of the figures shown by the luminous bands produced by transformer and coil discharges as shown in Figs. 7 and 8 depends upon the form and the magnitude of the current in the tube. This will be seen from the record of an oscilloscope as shown in the left hand side of the photographs. In a transformer discharge,

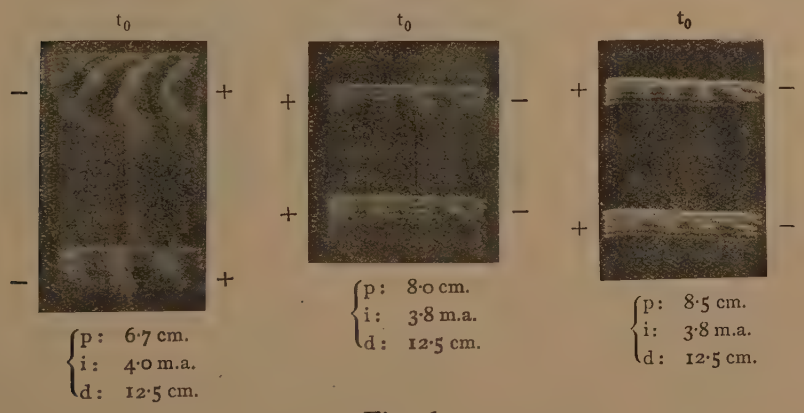


Fig. 6.

the current is initially zero, and the corresponding luminosity appears in the neighborhood of the anode, while in the case of the coil discharge the current is maximum at its start, giving the first luminosity at a place very near the cathode.

Thus the process of formation of the luminous patches at a relatively high pressure seems to be similar to that of striation in the

positive column. The only difference to be considered is that in the case of the positive column the distance between the striae is smaller than the mean free path of the gas molecules in the discharge tube, while in the case o the luminous patches their distance is generally greater than the free path.

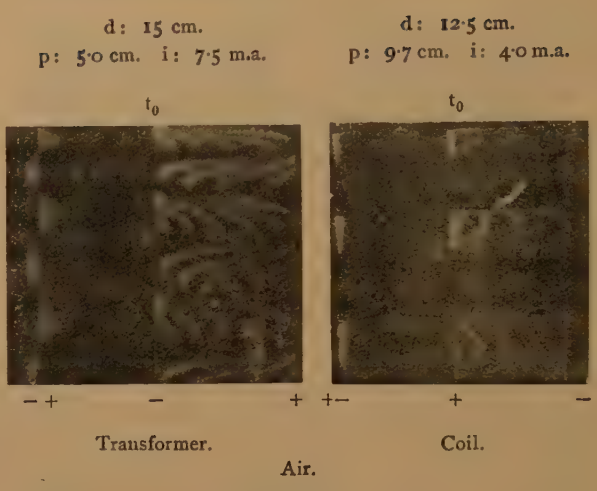


Fig. 7.

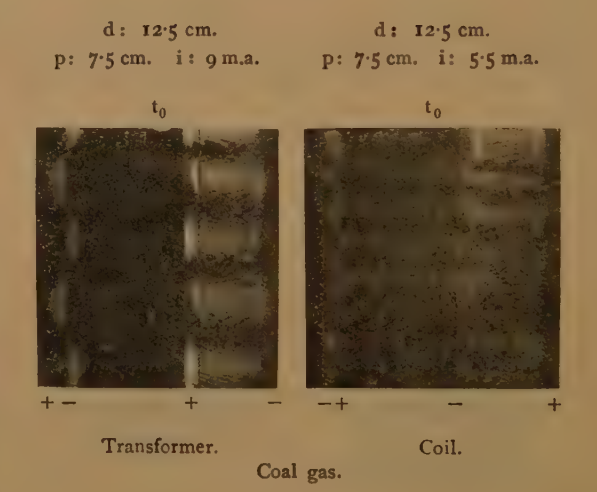


Fig. 8.

To sum up, the formation of the luminous bands may now be conveniently explained in the following way. As the potential difference acting on the terminals of the discharge tube is increased, positive ions bombarding the cathode will produce a number of electrons which, driven by an intense electric field in the neighborhood of the cathode, will acquire sufficient kinetic energy to make the gas luminous. By this process they lose their velocities, and consequently their path becomes dark. But they soon regain sufficient energy after having travelled through certain distances, and then they make the gas luminous again. This process will go on repeatedly giving alternately dark and bright spots. Thus for a single partial discharge the distance between bright and dark spots should be nearly equal, and it should increase as the potential differ-

ence between the terminals is lowered. This will be seen to be the case in our preceding photographs.

Summary

1. Electric discharges through tubes containing the air, hydrogen or coal gas at pressures of a few centimeters were examined with a falling plate camera, and the apparently unstriated discharges consisting of a number of luminous bands were studied.
2. An explanation of formation of these luminous bands was given.

It is here desirable to add the following. The present invetigation was started at the beginning of October, 1919 and finished at the end of April, 1920. While this paper was going through the press, Drs. Aston and Kikuchi's papers dealing with the similar subject appeared in the *Nature* [vol. 105, 633 (1920)] and also in the *Proceedings of the Royal Society of London* [A. vol. 98, 50 (1920)].

In conclusion, the writer wishes to express his hearty thanks to Prof. M. Kimura for his valuable suggestions he has given throughout the work.

Some Properties of Analytic Elements On And Outside Their Circles of Convergence

By

Toshizô Matsumoto

(Received December 21, 1920)

Introduction.

Consider a mathematical expression

$$F(z)\equiv c_0+c_1z+\ldots+c_nz^n+\ldots,$$

where z is a complex variable. We call it a *series* (more strictly a *power-series*). If $\lim_{n\to\infty} S_n(z)$, where

$$S_n(z)\equiv c_0+c_1z+\ldots+c_nz^n,$$

be finite and determinate, the series is convergent. When the limiting value is ∞ or indeterminate, the series is divergent. In the former case, put

$$S_n(z)\equiv A_n+iB_n.$$

Then the equality $\lim_{n\to\infty} S_n(z)=\infty$ means that $\lim_{n\to\infty}|A_n+iB_n|=\infty$; but not at the same time $\lim A_n=\infty$ and $\lim B_n=\infty$. Even when these limits are indeterminate, yet $S_n(z)$ may tend to ∞. Let us call the series *infinite* when both A_n and B_n tend to definite limiting values and $|A_n+B_n|$ tends to infinity. When at least one of A_n and B_n becomes indeterminate, we call the series *indeterminate*.

In the present paper, the intervals in which a power-series is convergent, infinite or indeterminate, are discussed. Next, it is proved that an analytic element outside its circle of convergence and in the part of the plane toward which the element may be continued, is

indeterminate. Thirdly, the relation between Leibnitz's theory and Borel's generalised sums are considered. Finally, some special cases of the theorems of Fabry and of Hadamard concerning singular points are viewed from our standpoint.

1. Consider a power-series

$$F(z)=c_0+c_1z+\ldots+c_nz^n+\ldots,$$

whose radius of convergence may in general be supposed to be unity. The other case is now out of question. If the series be convergent at a point, say $z=1$, on its circle of convergence, then the power-series $F(z)$, when z converges to unity, takes the value $F(1)$. Thus the extension of Abel's theorem is due to Stolz.[1] In his book,[2] Mr. Stolz proved furthermore that if the series be infinite at $z=1$, then the coefficients being supposed real, along the real axis of z-plane, $\lim_{x\to 1} F(x)=\infty$.

By aid of these theorems, we may easily conclude that if $F(z)$ may be continued over $z=1$, then the series $F(z)$ must be convergent or indeterminate at $z=1$. Put

$$z=x+iy, \qquad c_n=a_n+ib_n, \qquad n=0,\ 1,\ \ldots\ n,\ \ldots,$$
$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots,$$
$$f_1(x)=b_0+b_1x+\ldots+b_nx^n+\ldots,$$

and let

$$F(z|z_0)=P+iQ,$$

be the continuation of $F(z)$ over $x=1$, where z_0 is a point within the interval $(0, 1)$. If z varies along the x-axis, then in the vicinity of $z=1$, $(x<1)$, we have $f(x)=P$, $f_1(x)=Q$. Therefore, neither $f(1)$ nor $f_1(1)$ can be infinite. Suppose, for example, $f(1)=\infty$, then by Stolz's theorem

$$\lim_{x\to 1} f(x)=\infty.$$

Hence for any given positive number G, we have

$$f(x)>G,$$

provided x is sufficiently near to unity. On the other hand, since $F(z|z_0)$ is in the vicinity of $z=1$, is convergent, we may suppose G such that

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1 Goursat, *Cour d'Analyse* II (1918), p. 22.

2 Stolz-Gemeiner, *Funktionentheorie* (1905), p. 289.

$$|F(z|z_0)|< G.$$

This is absurd, and the same may be said for $f_1(x)$. Thus, *any power-series must be convergent or indeterminate at the points on its circle of convergence, over which the series may be continued. Therefore if* $F(1)$ *be infinite, then* $F(z)$ *can not be continued over* $z=1$.

2. For example, consider the series

$$F(z)=1-z+z^2-\dots,$$

whose radius of convergence is unity and the series may be continued over $z=1$. But for $z=1$,

$$F(1)=1-1+1-\dots,$$

as is well known, is indeterminate and several objections[1] are made against Euler's assumption. I shall present one:
Since, n being a positive integer,

$$\frac{1}{2}\int_{n\pi}^{(n+1)\pi}\sin x dx=(-1)^{n+1},$$

we have

$$I_\infty=\frac{1}{2}\int_0^\infty \sin x dx=\frac{1}{2}\int_0^\pi+\frac{1}{2}\int_\pi^{2\pi}+\dots$$
$$=1-1+1-1+\dots.$$

On the other hand, we have

$$I_\alpha=\frac{1}{2}\int_0^\alpha \sin x dx$$
$$=1-1+\dots+(-1)^{n-1}+\frac{1}{2}\int_{n\pi}^{\alpha}\sin x dx.$$

Putting $\alpha=n\pi+A$, where A is an angle less than $\frac{\pi}{2}$, we have

$$I_\alpha=\frac{1}{2}(1\pm\cos A).$$

So that for $\alpha\to\infty$, our series $F(1)$ would take any value in $(0, 1)$. Nevertheless for $|z|<1$,

$$F(z)=\frac{1}{1+z}.$$

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1 Borel, *Séries divergentes* (1901).

Therefore the analytic function defined by the analytic element $F(z)$ takes $\frac{1}{2}$ at $z=1$ and Euler's assumption is to some degree not absurd. A complete discussion will be given later.

3. In the following we shall show some theorems concerning indeterminate series.

Consider a series

$$S=a_0+a_1+\ldots+a_n+\ldots,$$

which is indeterminate. Construct two series

$$S'=a_0'+a_1'+\ldots+a_n'+\ldots,$$
$$S''=a_0''+a_1''+\ldots+a_n''+\ldots,$$

the first being made of all positive terms in S, taken in order, and the second, negative terms with signs changed. Then both series S' and S'' must diverge to infinity. Consequently *the number of variations of signs of each two successive terms of the series must be infinite.*

4. Next consider a power-series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots.$$

For simplicity we suppose this real. Put

$$S_n(x)=a_0+a_1x+\ldots+a_nx^n.$$

If the series be convergent in an interval, then in it

$$\lim_{n\to\infty} S_n(x)=\textit{finite and determinate},$$

which means that $S_n(x)$ tends to one and only one value, *independent of the mode of summation.* By our last expression, we mean to increase *n in any manner* and yet $S_n(x)$ converges to a number. But we do not mean to change the *order of the terms* of the series $f(x)$. On the contrary, if the series be indeterminate, then $\lim_{n\to\infty} S_n(x)$ depends upon the mode of summation.

Suppose for any value of n,

$$|S_n(x)|<G,$$

where G is a positive number, we write it as

$$|f(x)|<G,$$

and say that the series $f(x)$ is limited. *If $f(x)$ be limited at a point $x=x_0$, then the series $f(x)$ is convergent for $|x|<|x_0|$.* For the proof of this, consider the series

$$\sum_{n=0}^{\infty} S_n(x_0)\left(\frac{x}{x_0}\right)^n, \quad \text{for} \quad |x|<|x_0|.$$

Since by our assumption, all $S_n(x)$ are limited, the series is absolutely convergent. Therefore the product of the two series:

$$\left\{\sum_{n=0}^{\infty} S_n(x_0)\left(\frac{x}{x_0}\right)^n\right\}\left(1-\frac{x}{x_0}\right)$$

is also absolutely convergent, and hence the order of its terms may be changed. Hence it is equal to the series

$$\begin{aligned} S_0(x_0)+\left\{S_1(x_0)-S_0(x_0)\right\}\left(\frac{x}{x_0}\right)+\ldots+\left\{S_n(x_0)-S_{n-1}(x_0)\right\}\left(\frac{x}{x_0}\right)^n+\ldots \\ =a_0+a_1x+\ldots+a_nx^n+\ldots \\ =f(x). \end{aligned}$$

Thus $f(x)$ must be convergent. This is an example of the theorem that if all the terms of the series $f(x)$ be limited at $x=x_0$, then it is convergent for $|x|<|x_0|$.[1] Hence *if $f(x)$ be divergent in an interval (x_1, x_2) where $0<x_1<x_2$, then* it can not be limited within the interval. Moreover, since $f(x)$ can not be convergent for $x>x_2$, *it can not be limited in the interval* (x_1^*, x_2). x_1^* means the exclusion of the point $x=x_1$. The series $1-x+x^2-\ldots$ is such an example for the interval $(1^*, \infty)$.

Moreover, we may prove that if $f(x)$ be indeterminate in an interval (x_1, x_2), it must not be *limited in one way*, namely neither

$$f(x)>G, \quad \textit{nor} \quad f(x)<G'$$

may be true, where G and G' are certain numbers. We shall prove *e.g.*, the case where $f(x)>G$. In this inequality we may suppose $G>0$. For this inequality means that for any value of n, $S_n(x)>G$. Hence if G be negative, we have only to add a certain positive number to a_0. Consequently, we may suppose that all $S_n(x)$ are positive in the interval.

1 Goursat, *Cours d'analyse* I (1917). p. 446.

Now x_0 being a number in the interval (x_1, x_2),

$$S_n(x)=\sum_{h=0}^{n}\left\{S_h(x_0)-S_{h-1}(x_0)\right\}\left(\frac{x}{x_0}\right)^h$$

$$=\left\{\sum_{h=0}^{n-1}S_h(x_0)\left(\frac{x}{x_0}\right)^h\right\}\left(1-\frac{x}{x_0}\right)+S_n(x_0)\left(\frac{x}{x_0}\right)^n, \qquad (1)$$

where $S_{-1}(x_0)\equiv 0$.

Let x take any value within the interval (x_1, x_0). To prove our proposition, three cases may occur: 1st. If the series

$$\sum_{h=0}^{\infty}S_h(x_0)\left(\frac{x}{x_0}\right)^h$$

be convergent, then we must have

$$\lim_{n\to\infty}S_n(x_0)\left(\frac{x}{x_0}\right)^n=0.$$

Therefore by aid of the equality (1), $f(x)$ must be convergent, and this is against our hypothesis that the series $f(x)$ is indeterminate. 2nd. If the series $\Sigma S_h(x_0)\left(\frac{x}{x_0}\right)^h$ be determinate and infinite, then since, by the preceding equality (1),

$$S_n(x)>\left\{\sum_{h=0}^{n-1}S_h(x_0)\left(\frac{x}{x_0}\right)^h\right\}\left(1-\frac{x}{x_0}\right). \qquad (2)$$

Hence $f(x)$ must be divergent and infinite. This again contradicts our hypothesis. 3rd. The only remaining possible case is that the series

$$\sum_{h=0}^{\infty}S_h(x_0)\left(\frac{x}{x_0}\right)^h$$

should be indeterminate. But if so, by the theorem of §3, the variations of the signs of the series must be infinite. But this is contrary to the assumption that

$$\text{all } S_n(x)>G>0.$$

Hence this assumption cannot but be absurd. Since $x_2-x_0>0$ may be however small, $f(x)$, *in the interval, can not be limited in both ways*. The discussion is unvalid at $x=x_1$. Moreover, suppose that *at a point* $x=x_0$, $(x_1<x_0<x_2)$, $f(x_0)$ be limited in one way *e.g.*,

$f(x_0)>G>0$, then as we have just proved, within the interval (x_1, x_0), it must give subintervals in which $f(x)$ is not limited in either way. Let x be such a point in one of them. Then by aid of the inequality (2), $f(x)$ must be limited in the negative way. This is absurd. Therefore we may conclude these as follows:

If the series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots$$

be divergent and indeterminate in the interval (x_1, x_2), *then the series* $f(x)$ *can not be limited in either way at any point in the interval* (x_1^*, x_2^*), $x_1>0$. *Therefore if* $f(x)$ *be limited only in one way, then it must diverge to infinity.* This result may be extended to the field of complex numbers.

Remark 1. As we have proved, an indeterminate power-series can not be limited, *i.e.*, $S_n(x)$ may take, at general points, values greater than any positive number, respectively any number less than any negative value. But for general functions, this is not true. For example $f(x)=\sin\frac{1}{x}$ is indeterminate for $x\to 0$, but it is not greater than unity, *i.e.*, limited in both ways.

Remark 2. Though an indeterminate power-series can not be limited in both ways, it may at certain points take finite determinate values for certain modes of summations. For example let $f(x)$ be

$$f(x)=x_0-\left(1+\frac{1}{1!}\right)x-x_0x^2+\left(1+\frac{1}{3!}\right)x^3+\ldots$$
$$+(-1)^nx_0x^{2n}+(-1)^{n+1}\left(1+\frac{1}{(2n+1)!}\right)x^{2n+1}\pm\ldots,$$

where x_0 is a number greater than unity. The radius of convergence is unity. This series may be continued over $x=1$; for if $|x|<1$, then clearly

$$f(x)=\frac{x_0-x}{1+x^2}-\sin x.$$

Thus $f(x)$ is a sum of functions which may be continued over $x=1$. Hence it may be continued. If at $x=x_0$, we sum up $f(x_0)$ by two successive terms from the beginning, we heve

$$f(x_0)=-\sin x_0.$$

On the contrary

$$\lim_{n\to\infty} S_{2n}=\pm\infty.$$

From the theorem of § 1, we naturally come to inquire whether a power-series be indeterminate outside the arc of its circle of convergence over which the series may be continued. For a special case it is clear. For example, the series

$$F(z)=1-z+z^2-\dots$$

may be continued over $z=1$ and is indeterminate for $z\geq 1$. We shall enter into the discussion of the general case where the answer is affirmative.

5. If a convergent power-series

$$f(x)=a_0+a_1x+\dots+a_nx^n+\dots$$

whose radius of convergence is unity, be determinate and infinite at $x=x_0$, $(x_0>1)$, then for $1<x\leq x_0$ $f(x)$ must also be infinite. For proof of this, consider as usual the identity

$$\begin{aligned}S_n(x)&\equiv a_0+a_1x+\dots+a_nx^n\\&\equiv\left\{S_0(x_0)+S_1(x_0)\left(\frac{x}{x_0}\right)+\dots+S_{n-1}(x_0)\left(\frac{x}{x_0}\right)^{n-1}\right\}\\&\qquad\times\left(1-\frac{x}{x_0}\right)+S_n(x_0)\left(\frac{x}{x_0}\right)^n.\end{aligned}$$

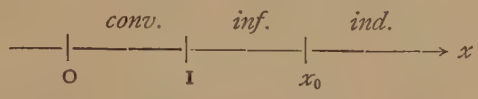
1°. If the series $\Sigma S_h(x_0)\left(\frac{x}{x_0}\right)^h$ be convergent, then since

$$\lim_{n\to\infty}S_n(x_0)\left(\frac{x}{x_0}\right)^n=0,$$

$S_n(x)$ must converge to a number, and this is impossible. 2°. The series $\Sigma S_h(x_0)\left(\frac{x}{x_0}\right)^h$ can not be indeterminate, since the variations of signs are finite. 3°. Consequently it must diverge to infinity. So that the series $f(x)$ is infinite in $1<x\leq x_0$. This is true also for $f(x_0)=-\infty$.

From this result, it follows that if $f(x)$ be indeterminate for $x'>x_0$, then it must be so for any $x>x_0$. If not, it gives at least a point $x'>x_0$, for which $f(x')$ is ∞, (or $-\infty$). Hence for any $x<x'$, $f(x)$ must be ∞ (or $-\infty$) which is contrary to our assumption. Thus *the points $x\geq 0$, are divided, if possible, into three parts:* 1°. $0\leq x<1$, 2°. $1<x<x_0$, 3°. $x_0<x$. *In the first interval $f(x)$ is convergent;*

in the second it is infinite; and in the third, it is indeterminate (by the preceding theorem, it oscillates between $-\infty, +\infty$).



6. Let us give an example. In the formula

$$S_n(x)=\left(1-\frac{x}{x_0}\right)\left\{S_0(x_0)+S_1(x_0)+\ldots+S_{n-1}(x_0)\left(\frac{x}{x_0}\right)^{n-1}\right\}+S_n(x_0)\left(\frac{x}{x_0}\right)^n,$$

put

$$S_h(x_0)\equiv s_h, \qquad h=0, 1, 2, \ldots, n.$$

Then for $x=2x_0$, we have

$$S_n(2x_0)=-(s_0+2s_1+\ldots+2^{n-1}s_{n-1})+2^n s_n.$$

Assume

$$s_{2h}=2^{2h+1}, \quad s_{2h+1}=2^{2h}, \quad h=0, 1, 2, \ldots, n.$$

Since

$$2^{2h}s_{2h}+2^{2h+1}s_{2h+1}=2\cdot 2^{4h+1},$$

we have for $n=2m$,

$$S_{2m}(2x_0)=-2(2+2^5+\ldots+4^{5m-3})+2^{4m+1}$$
$$=2^{4m+1}\left(1-\frac{2}{15}\right)+\frac{4}{15}>0.$$

For $n=2m+1$, we have

$$S_{2m+1}(2x_0)=-4\frac{2^{4m}-1}{15}<0.$$

Hence for $m\to\infty$,

$$\lim_{m\to\infty} S_{2m}(2x_0)=+\infty,$$
$$\lim_{m\to\infty} S_{2m+1}(2x_0)=-\infty,$$

and consequently our series $f(x)$ for $x=2x_0$ is indeterminate and not limited in both ways.

Now to find the concrete form of the series, we must solve the following equations with an infinite number of unknowns:

$$s_0=a_0,$$
$$s_1=a_0+a_1x_0,$$
$$s_2=a_0+a_1x_0+a_2x_0^2,$$
$$\ldots\ldots\ldots\ldots$$

$$s_n = a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_n x_0^n,$$

.....

This is a special case of the systems which are treated in Volterra's equations.[1] For our case, the treatment is easy.
We have

$$a_0 = s_0,$$

$$a_1 = \frac{s_1 - s_0}{x_0},$$

.....

$$a_n = \frac{s_n - s_{n-1}}{x_0^n},$$

.....

Therefore substituting the values of s_n, we have

$$a_0 = 2,$$

$$a_{2h} = \frac{2^{2h+1} - 2^{2(h-1)}}{x_0^{2h}} = \frac{7}{4}\frac{2^{2h}}{x_0^{2h}},$$

$$a_{2h+1} = -\frac{1}{2}\frac{2^{2h+1}}{x_0^{2h+1}}.$$

The required series is

$$f(x) = 2 - \frac{1}{2}\left(\frac{2}{x_0}\right)x + \frac{7}{4}\left(\frac{2}{x_0}\right)^2 x^2 - \frac{1}{2}\left(\frac{2}{x_0}\right)^3 x^3 + \ldots.$$

Specially for $x_0 = 2$, we have

$$f(x) = 2 - \frac{1}{2}x + \frac{7}{4}x^2 - \frac{1}{2}x^3 + \ldots, \qquad (1)$$

whose radius of convergence is unity and

$$f(1) = \infty, \qquad f(2 \times 2) = \pm\infty.$$

The point which divides the intervals where the series $f(x)$ becomes infinite, respectively, indeterminate is found to be $\frac{7}{2}$. For at this point

$$a_{2h}x^{2h} + a_{2h+1}x^{2h+1} = \left(\frac{7}{4} - \frac{1}{2}\frac{7}{2}\right)\left(\frac{7}{2}\right)^{2h} = 0,$$

$$a_{2h-1}x^{2h-1} + a_{2h}x^{2h} = \left(-\frac{1}{2} + \frac{7}{4}\frac{7}{2}\right)\left(\frac{7}{2}\right)^{2h-1} > 0.$$

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1 Volterra, *Leçons sur les Équations intégrales et ...*, (1913), p. 40.

Thus $S_n\left(\frac{7}{2}\right)$ is limited in one way. Therefore by the theorem of § 5, for x just less than $\frac{7}{2}$, the series must be determinately infinite. Concluding these, for our series (1), the whole interval of x is divided into four parts:

$$\underset{x\leq}{+inf.} \quad \underset{-1<x<+1}{conv.} \quad \underset{+1\leq x<3\cdot5}{+inb.} \quad \underset{3\cdot5<x}{ind.}$$

At $x=3\cdot5$, $f(x)$ oscillates between 0 and ∞.

7. In § 1, we have proved, by aid of Stolz's theorem, that a power-series $F(z)$ which may be continued over $z=1$, must be convergent or indeterminate at $z=1$. If it be indeterminate at $z=1$, then by § 5, it must be indeterminate for $z>1$ along the real axis of the z-plane. But if it be convergent at $z=1$, whether it is indeterminate for $z>1$ is not known. In the following we shall prove it. As before, instead of considering $F(z)$, we discuss its real or imaginary part for real values of z which is divergent for $z>1$, namely

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots.$$

Since $F(z)$, by our assumption, may be continued over $z=1$, its real and imaginary parts may also be continued over $x=1$. Therefore $f(x)$ may be continued over $x=1$.

8. For the general proof, we take a quite different method which is nothing but to connect three theorems proved by Borel, Hardy and Vivanti. In the following the process is briefly stated.

Since the series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots$$

may be continued over $x=1$, the analytic function defined by $f(x)$ is holomorph about $x=1$. Now draw two concentric circles, center at the point $x=\frac{1}{2}$. Let the radius of the inner circle be $\frac{1}{2}$. Make that of the outer one just greater than this; so that in and on the circle, the analytic function may be holomorph. This circle intersects the real axis at O' and A. Let the circle be called C. By Cauchy's theorem for the analytic function $f(x)$, we have

$$a_n=\frac{1}{2\pi i}\int_C\frac{f(z)}{z^{n+1}}dz, \qquad n=0, 1, 2, \cdots$$

Consider the associated function of $f(x)$:

$$A(t)=a_0+a_1x\frac{t}{1!}+\ldots+a_nx^n\frac{t^n}{n!}+\cdots,$$

where t is a real variable. This function is clearly an integral function. Putting the values of a_n, we have for any value of t,

$$A(t)=\frac{1}{2\pi i}\int_C\frac{f(x)}{z}e^{\frac{zt}{z}}dz.$$

So we have

$$e^{-t}A(t)=\frac{1}{2\pi i}\int_C\frac{f(z)}{z}e^{t\left(\frac{x}{z}-1\right)}dz,$$

$$e^{-t}A^{(h)}(t)=\frac{1}{2\pi i}\int_C\frac{f(z)x^h}{z^{h+1}}e^{t\left(\frac{x}{z}-1\right)}dz.$$

Therefore for any x of a certain ellipse whose major axis is $O'A$, our series $f(x)$, is absolutely summarble. Thus the polygon of summability may be determined. These are due to Mr. Borel.[1]

Now after Mr. Hardy[2], writing for simplicity,

$$u_n=a_nx^n,\ s_n=u_0+u_1+\cdots+u_n,\ n=0,\ 1.\ 2,\ \ldots,$$

we have

$$e^{-t}\sum_{n=0}^{\infty}s_n\frac{t^n}{n!}-u_0=\int_0^t e^{-t}\sum_{n=0}^{\infty}\frac{u_{n+1}t^n}{n!}dt.$$

But the series $u_0+u_1+\ldots+u_n+\ldots$ is, as we have seen, summable. Therefore

$$\lim_{t\to\infty}e^{-t}\sum_{n=0}^{\infty}s_n\frac{t^n}{n!}$$

is finite and determinate; so that our series Σu_n admits both of Borel's definitions of summability.

Next suppose our series $f(x)$ *i.e.*, Σu_n be ∞, then we can determine a positive number N, such that given a number however great G,

$$s_n>G,\quad \text{for any}\quad n>N.$$

Therefore

$$e^{-t}\sum_{n=0}^{\infty}s_n\frac{t^n}{n!}>e^{-t}\sum_{n=0}^{N-1}(s_n-G)\frac{t^n}{n!}+G,$$

and hence

---

1 Borel, *loc. cit.*, p. 123.

2 Hardy, *Quarterly J.* (1904), p. 34.

$$\lim_{t\to\infty} e^{-t}\sum_{n=0}^{\infty} s_n \frac{t^n}{n!}=\infty.$$

This proof is due to Mr. Vivanti[1]. This result is contradictory. Thus our series must be indeterminate. Now by aid of the theorem of § 5, our proposition is true:

When a power-series, whose radius of convergence is unity, $f(x)=a_0+a_1x+\cdots+a_nx^n+\ldots$ *may be continued over* $x=1$, *then it must be indeterminate for all points* $x>1$. *For* $x=1$, *the series may sometimes be convergent or sometimes be indeterminate.* Moreover when $f(x)$ may be continued over $x=1$, this is also the case for any derivatives of $f(x)$, Therefore, *at the same time, all derivatives of* $f(x)$ *are indeterminate, for* $x>1$.

9. The above theorem is a necessary condition of the analytic continuation, but not clearly sufficient. Or, take, for example, Mr. Fredholm's series, changing a into $-a$,

$$f(x)=1-ax+a^2x^2-\ldots+(-1)^n a^n x^{n^2}\pm\ldots,$$

where a is a positive number. The circle of convergence is the natural limit of the function. But for $x>1$,

$$\frac{a^{n+1}x^{(n+1)^2}}{a^n x^{n^2}}=ax^{2n+1}>1,$$

provided n is sufficiently great. Therefore

$$(1-ax)+(a^2x^{2^2}-a^3x^{3^2})+\ldots=-\infty,$$
$$1-(ax-a^2x^{2^2})-(a^3x^{3^2}-a^4x^{4^2})-\ldots=+\infty,$$

so that $f(x)$ is indeterminate for $x>1$; likewise for its derivatives.

10. In remark 2, § 4, we have given a power-series which takes a value by a certain mode of summation. From that series, we obtain a new one,

$$f(x)=\frac{x_0-x}{1+x^2}$$

$$=x_0-x-x_0x^2+x^3+\ldots+(-1)^n x_0x^{2n}+(-1)^{n+1}x^{2n+1}\pm\ldots.$$

We see that.

$$\lim_{n\to\infty} S_{2n+1}(x_0)=S_{2n+1}(x_0)=0.$$

---

1 Vivanti, *Theorie der Funktionen* (1906), p. 329.

Here we notice that o is the value of the analytic function $f(x)$. Therefore we naturally come to conjecture that such would be a general property. But this is at once proved to be wrong. For, consider the series

$$f(x)=1-x+x^2-\ldots+(-1)^n x^n\pm\ldots .$$

For this series $S_{2n+1}(1)=0$. But the value of the analytic function at $x=1$ is $\frac{1}{2}$, different from zero. We notice that o is the lower limit of $S_n(1)$. This is not the case for the preceding series. The second conjecture is that when the limit of $S_n(x_0)$ at some point $x=x_0$ for a certain mode of summation converges to a number different from the upper respectively lower limit of $S_n(x_0)$, that limit would be the value of the analytic function at $x=x_0$. This conjecture is also destroyed by the consideration of the following series;

$$1+\cos\theta+\cos 2\theta+\ldots+\cos n\theta+\ldots .$$

This series is divergent and θ being not equal to $2m\pi$, its Borel's integral sum is easily found to be $\frac{1}{2}$[1]. Let us consider a power-series

$$f(x)=1+x\cos\theta+x^2\cos 2\theta+\ldots+x^n\cos n\theta+\ldots ,$$

which is convergent within the circle of radius unity. At the point $x=1$, it is divergent. Now put

$$S_n(1)=1+\cos\theta+\cos 2\theta+\ldots+\cos n\theta,$$

Then

$$2\,S_n(1)=1+\frac{\sin\frac{2n+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

Specially consider the case $\theta=\frac{\pi}{4}$, then

$$2\,S_n(1)=1+\frac{\sin(2n+1)\frac{\pi}{8}}{\sin\frac{\pi}{8}},$$

$$\begin{aligned}
&=1+1, && n\equiv 0,\ 3 \pmod 8,\\
&=1+\cot\frac{\pi}{8}, && \equiv 2,\ 1 \quad ,, \quad ,\\
&=1-1, && \equiv 4,\ 7 \quad ,, \quad ,\\
&=1-\cot\frac{\pi}{8}, && \equiv 6,\ 5 \quad ,, \quad .
\end{aligned}$$

[1] Bromwich, *Theory of infinite series* (1908), p. 275.

Therefore for $n=8m+4$,

$$S_n=\lim S_n=0.$$

Since $2<\cot\frac{\pi}{8}<3$, this value lies between the upper and the lower limit of the sums.

The a nalytic function can easily be found to be

$$f(x)=\frac{\sqrt{2}+x-x^3}{\sqrt{2}(1+x^4)},\quad f(1)=\frac{1}{2}.$$

Hence for our mode of summation

$$f(1)\neq\lim_{n\to\infty}S_n(1).$$

The third conjecture is that when S_n tend to several values, among them there should occur the value of the analytic function. This is also fatal. For by aid of the last example, If $S_n(1)$ tend to $\frac{1}{2}$, by a certain mode of summation, then

$$\lim_{n\to\infty}\sin(2n+1)\frac{\pi}{8}=0.$$

And this is impossible. All the conjectures are proved to be wrong on the circle of convergence.

11. Now we notice in the preceding problem that

$$\lim_{k\to\infty}S_{8k+h}(1)=\textit{finite and determinate},\ h=0,1,2,\ldots,7,$$

and their arithmetical mean $\frac{1}{2}$ is nothing bnt the value of the analytic function at $x=1$. Consequently we come to the fourth *conjecture*. Namely, *given a divergent series*

$$S=a_0+a_1+\ldots+a_n+\cdots,$$

if

$$\lim_{k\to\infty}S_{km+h}=S^{(h)},\qquad h=0,1,\ldots,m-1,$$

be all finite and determinate, then

$$\frac{1}{m}\left(S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}\right)$$

is equal to the generalised sum of the given series. This supposition is true:

At first we shall consider Cèsaro's generalisation. Since the sequence of numbers

$$S_h, S_{m+h}, \ldots, S^{km+h}, \ldots$$

converges to $S^{(h)}$, we have

$$\lim_{k\to\infty} \frac{S_h+S_{m+h}+\ldots+S_{km+h}}{k} = S^{(h)}$$

$$= \lim_{k\to\infty} \frac{S_h+S_{m+h}+\ldots+S_{km+h}}{km+m-1} \cdot \frac{km+m-1}{k}.$$

$$\therefore \lim_{k\to\infty} \frac{S_h+S_{m+h}+\ldots+S_{km+h}}{km+m-1} = \frac{S^{(h)}}{m}, \quad h=0, 1, \ldots, m-1.$$

Adding these m equations side by side, we may conclude that

$$\lim_{k\to\infty} \frac{S_0+S_1+\ldots+S_{km+m-1}}{km+n-1} = \frac{1}{m}(S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}).$$

Hence, in general,

$$\lim_{n\to\infty} \frac{S_0+S_1+\ldots+S_n}{n} = \frac{1}{m}(S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}). \quad (1)$$

Q. E. D.

And our series is *simply indeterminate.*

Next, we consider Borel's generalisation. For this purpose consider the limits

$$\lim_{t\to\infty} \frac{E_h(t)}{e^t}, \qquad h=0, 1, \ldots, m-1,$$

where

$$E_h(t) \equiv \frac{t^h}{h!} + \frac{t^{m+h}}{(m+h)!} + \ldots + \frac{t^{km+h}}{(km+h)!} + \ldots.$$

The functions $E_h(t)$ satisfy the differential equation

$$\frac{d^m E}{dt^m} = E.$$

Therefore let $r_0(=1), r_1, \ldots, r_{m-1}$ be the roots of the characteristic equation

$$r^m = 1,$$

then we have

$$E_h(t) = C_{h0}e^{r_0 t} + C_{h1}e^{r_1 t} + \ldots + C_{hm-1}e^{r_{m-1}t}, \quad (2)$$

$$h=0, 1, \ldots, m-1,$$

where $C's$ are the integration-constants. These constants are to be so determined that

$$C_{h0}+C_{h1}+\ldots+C_{hm-1} = 0,$$

$$C_{h0}r_0+C_{h1}r_1+\ldots+C_{hm-1}r_{m-1} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$C_{h0}r_0^h + C_{h1}r_1^h + \ldots + C_{hm-1}r_{m-1}^h = 1,$$

$$\cdots\cdots\cdots\cdots$$

$$C_{h0}r_0^{m-1} + C_{h1}r_1^{m-1} + \ldots + C_{hm-1}r_{m-1}^{m-1} = 0.$$

To find C_{h0}, its denominator is equal to

$$\Delta \equiv \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_0 & r_1 & \ldots & r_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ r_0^{m-1} & r_1^{m-1} & \ldots & r_{m-1}^{m-1} \end{vmatrix} = \pm (r_0 - r_1)(r_0 - r_2) \ldots\ldots (r_0 - r_{m-1})$$
$$(r_1 - r_2) \ldots\ldots (r_1 - r_{m-1})$$
$$\ldots\ldots \quad \ldots\ldots$$
$$(r_{m-2} - r_{m-1}).$$

The numerator is equal to

$$\Delta_{h0} \equiv \begin{vmatrix} 0 & 1 & \ldots & 1 \\ 0 & r_1 & \ldots & r_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & r_1^h & \ldots & r_{m-1}^h \\ \cdots & \cdots & \cdots & \cdots \\ 0 & r_1^{m-1} & \ldots & r_{m-1}^{m-1} \end{vmatrix} = \pm \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_{m-1} \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{h-1} & r_2^{h-1} & \ldots & r_{m-1}^{h-1} \\ r_1^{h+1} & r_2^{h+1} & \ldots & r_{m-1}^{h+1} \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{m-1} & r_2^{m-1} & \ldots & r_{m-1}^{m-1} \end{vmatrix}$$
$$= M(r_1 - r_2)(r_1 - r_3) \ldots (r_1 - r_{m-1})$$
$$(r_2 - r_3) \ldots (r_2 - r_{m-1})$$
$$\cdots\cdots\cdots\cdots$$
$$(r_{m-2} - r_{m-1}),$$

where M in a certain factor to be determined. Since

$$r^m - 1 = (r-1)(r^{m-1} + r^{m-2} + \ldots + 1)$$
$$= (r-1)(r-r_1)(r-r_2) \ldots (r-r_{m-1}),$$

for $r = r_0 (= 1)$, we have

$$(r_0 - r_1)(r_0 - r_2) \ldots (r_0 - r_{m-1}) = m. \qquad (3)$$

On the other hand, with respect to the equation

$$1 + r + \ldots + r^h + \ldots + r^{m-1} = 0, \qquad (4)$$

we have the relations

$$1 + r_1 + \ldots + r_1^h + \ldots + r_1^{m-1} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$1+r_{m-1}+\ldots+r_{m-1}^{h}+\ldots+r_{m-1}^{m-1}=0.$$

Hence we have

$$\begin{vmatrix} 1 & r & \ldots & r^{h} & \ldots & r^{m-1} \\ 1 & r_1 & \ldots & r_1^{h} & \ldots & r_1^{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & r_{m-1} & \ldots & r_{m-1}^{h} & \ldots & r_{m-1}^{m-1} \end{vmatrix} = 0.$$

This equation must be identical with (4). The coefficients of r^{m-1} is

$$\pm(r_1-r_2)(r_1-r_3)\ldots(r_1-r_{m-1})$$
$$(r_2-r_3)\ldots(r_2-r_{m-1})$$
$$\cdots\cdots\cdots\cdots$$
$$(r_{m-2}-r_{m-1}),$$

That of r^{h} is nothing but $\pm\Delta_{h0}$. Hence, noticing that the coefficients of the equation (4) are all unity, we have

$$M=\pm 1.$$

Hence by (3) we have

$$C_{h0}=\frac{\Delta_{h0}}{\Delta}=\pm\frac{1}{m}, \qquad h=0, 1, \ldots, m-1.$$

On the other hand, since the real parts of $r's$ except $r_0=1$ are all less than unity, we have

$$\lim_{t\to\infty}\frac{e^{r_h t}}{e^{t}}=0, \qquad h=1, 2, \ldots, n-1.$$

Therefore we have from (2)

$$\lim_{t\to\infty}\frac{E_h(t)}{e^{t}}=C_{h0}=\frac{1}{m}, \qquad h=0, 1, 2, \ldots, m-1. \qquad (5)$$

(This is clear, since we must take the + sign.)
Now since the sequence $S_h, S_{m+h}, \ldots, S_{km+h}, \ldots$ converges to $S^{(h)}$, we have

$$\lim_{t\to\infty}\frac{S_h\frac{t^{h}}{h!}+S_{m+h}\frac{t^{m+h}}{(m+h)!}+\ldots+S_{km+h}\frac{t^{km+h}}{(km+h)!}+\ldots}{E_h(t)}=S^{(h)},$$
$$h=0, 1, \ldots, m-1.$$

Whence by aid of (5),

$$\lim_{t\to\infty}\frac{S_h\frac{t^h}{h!}+S_{m+h}\frac{t^{m+h}}{(m+h)!}+\ldots+S_{km+h}\frac{t^{km+h}}{(km+h)!}+\ldots}{e^t}$$

$$=\lim_{t\to\infty}S^{(h)}\frac{E_h(t)}{e^t}=\frac{S^{(h)}}{m}.$$

So we reach the result

$$\lim_{t\to\infty}\frac{S_0+S_1\frac{t}{1!}+\ldots+S_n\frac{t^n}{n!}+\ldots}{e^t}$$

$$=\frac{S^{(0)}+S^{(1)}+\ldots+S^{(m-1)}}{m}. \qquad Q.E.D. \tag{6}$$

Applying this result to the analytic function defined by

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots$$

which as often said may be continued over $x=1$, *the value of the analytic function* $f(x)$ *at* $=1$ *is equal to the arithmetical mean of* $S^{(0)}+S^{(1)}+\cdots+S^{(m-1)}$, where

$$\lim_{k\to\infty}S_{km+h}=S^{(h)}, \qquad h=0, 1, \ldots, m-1,$$

are all supposed to be finite and determinate.

Outside the circle of convergence $x>1$, these limits can not all be finite and the theorem is clearly untrue.

12. After the fourth conjecture has been proved, we may give a complete discussion according to the series

$$S=1-1+1-1+\ldots. \tag{1}$$

After Mr. Borel[1], Euler and others took $\frac{1}{2}$ as the value of the series. Now, by aid of § 11, $\frac{1}{2}$ the arithmetical mean is equal to the generalised sums. If we consider the function defined by the series.

$$f(x)=1-x+x^2-\ldots,$$

the arithmetical mean $\frac{1}{2}$ is equal to the value of Borel's integral sum at $x=1$, which is the value of the analytic function defined by

1 Borel, *loc. cit.* pp. 4–7.

$f(x)$. That Leibnitz[1] gave $\frac{1}{2}$ as the value of the series from the *standpoints of the theory of probability*, corresponds by our side, to the equality (1) of the last article.

For Lagrange's serier[1].

$$f(x)=1+0+0-x^3+0+x^5+0+0-x^8+0+x^{10}+0+\dots,$$

using the same notations as before for $x=1$,

$$S_{5k+h}=1, 1, 1, 0, 0, \quad \text{according as } h=0, 1, 2, 3, 4.$$

Hence by our theorem, $\frac{3}{5}$ is the value of the analytic function

$$f(x)=\frac{1-x^3}{1-x^5}=\frac{x^2+x+1}{x^4+x^3+x^2+x+1}$$

at $x=1$. To give another value does not rest upon any theoretical standpoint.

Returning to the series (1) the discovery of Borel's sum verifies the fact that Leibnitz's theory is not incorrect. But whether this is generally true or not, is not known. By our proof, *the gap between Leibnitz's theory and Borel's generalisation has been crossed.* At the same time, Pringsheim's objection looses any value as Mr. Borel has already declared.[2]

We shall add another example.

$$S=0+\sin\theta+\dots+\sin n\theta+\dots,$$

$$2S=\cot\frac{\theta}{2}-\frac{\cos\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

For simplicity put $\theta=\frac{\pi}{4}$, then

$$2S_n=0, \qquad n\equiv 0, 7 \pmod 8,$$

$$=\cot\frac{\pi}{8}-1, \qquad \equiv 1, 6 \quad ,,$$

$$=\cot\frac{\pi}{8}+1, \qquad \equiv 2, 5 \quad ,,$$

---

1 Borel, *loc. cit.*, p.p. 4-7.

2 Borel, *loc. cit.*, p. 7.

$$= 2 \cot \frac{\pi}{8}, \qquad \equiv 3, 4 \quad ,, \quad .$$

Hence the generalised sum is equal to $\frac{1}{2} \cot \frac{\pi}{8}$, which can easily be calculated by Borel's integral.[1]

13. Let us return to the discussion of singular points on the circle of convergence.

If the series

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots ,$$

may be continued over $x = 1$, then as we have proved in § 8, $f(x)$, for $x > 1$, (sometimes $x = 1$ inclusive) must be indeterminate. Therefore by § 3, the number of variations of signs in $f(x)$ must be infinite. Consequently we have the following result:

Given a series of real coefficients

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n + \ldots ,$$

if between the coefficients

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1,$$

then the point $x = 1$ *must be a singular point of the analytic function defined by* $f(x)$. *If the radius of the convergence be unity, and if*

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} > 0,$$

the point $x = 1$ *is a singular point of the function. If* $a_n = 0$, *suppose* a_{n+1} *to be the consecutive of* a_{n-1}. This is a special case of Fabry's theorem.[2] A special case of his theorem is that given a power-series

$$F(x) = c_0 + c_1 z + \ldots + c_n z^n + \ldots ,$$

if between the coefficients

$$\lim_{n \to \infty} \frac{c_n}{c_{n+1}} = 1 ,$$

hen $z = 1$ is a singular point of the analytic function defined by $F(z)$. But our result is wider than this special case. For consider the series

$$F(x) = (1 + i) + (1 - i) z + \ldots + (1 + (-1)^n i) z^n + \ldots .$$

---

1 Bromwich, *loc. cit.*, p. 275.

2 Hadamard, *La Série de Taylor*, (1901) p. 25.

Then

$$\lim_{n\to\infty}\frac{c_n}{c_{n+1}}=\pm i.$$

Yet

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1,$$

and $z=1$ is a singular point.

If, moreover,

$$|a_n-A|<\varepsilon^n,$$

where $\varepsilon<1$, then at the point $x=1$, $f(x)$ becomes infinity. This point is a pole of the first order. For

$$\left|\sum_{n=m}^{m+h}a_nx^n-A\sum_{n=m}^{m+h}x^n\right|<\sum_{n=m}^{m+h}(\varepsilon x)^n,$$

in the vicinity of the point $x=1$. This implies the special case of Mr. Hadamard's theorem.[1]

From these considerations, we notice that several researches are intimately related to the *simple* property that the analytic element $F(z)$ must be indeterminate outside the circle of convergence to which it may be continued.

14. If in the preceding conditions, A be zero, *i.e.*, when

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1,\qquad |a_n|<\varepsilon_n,$$

where $\Sigma\varepsilon_n$ is convergent, then the point $x=1$ is not to be a pole and is either an *essential singularity* or a *branch-point* of the analytic function defined by $f(x)$. For by the second condition $f(1)$ is convergent. While by the first condition, the point must be a singular point. Examples are these:

1° $$f(x)=1+\frac{x^2}{1^2}+\frac{x^{2^2}}{2^2}+\ldots+\frac{x^{2^n}}{n^2}+\ldots.$$

2° $$f(x)=1-\frac{x}{2}-\frac{1}{2!}\left(\frac{x}{2}\right)^2-\ldots-\frac{1\cdot3\cdot5\cdot\ldots(2n-3)}{n!}\left(\frac{x}{2}\right)^n-\ldots.$$

The first series has its circle of convergence as its natural limits.

---

[1] Hadamard, *loc. cit.*, p. 39.

[2] Hadamard, *loc. cit.*, p. 32.

For the second which is an element of $(1-x)^{\frac{1}{2}}$, $x=1$ is the branch-point. Both are convergent at $x=1$.

15. At the end we shall shortly consider some divergent double series.

Consider as usual the series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots.$$

Suppose it may be continued over $x=1$, then taking a point $x=a$ within the interval (0, 1), the series

$$f(x|a)=f(a)+\frac{(x-a)}{1!}f'(a)+\ldots+\frac{(x-x)^n}{n!}f^{(n)}(a)+\ldots$$

is convergent for certain values of x. This may be stated as follows

The double-series, writing $x-a\equiv\xi$,

$$\begin{aligned} S= \quad & a_0 \\ & +a_1a+a_1\xi \\ & +a_2a^2+2a_2a\xi+\xi^2 \\ & +\ldots\ldots\ldots \\ & +a_na^n+na_na^{n-1}\xi+\ldots+\xi^n \\ & +\ldots\ldots\ldots \end{aligned}$$

is convergent, if we sum up first by columns and then by rows. But if we sum up first by rows and then by columns, the sum is indeterminate. Pushing such considerations further, we may give several cases.

INDEX OF AUTHORS.

PAGE.

Ando, K.,—see Y. Osaka. 371
Arakatsu, B.,—"On the Limiting Size of Colloidal Particles in a Browian Motion"... 179
—— —— see M. Ishino. 355
Fukuda, M.,—see B. Arakatsu... 179
—— —— Optical Properties of Japanese Millet Jelly. 183
—— —— The Absorption of Light by Sulphur at various Temperatures. 351
—— —— see M. Kimura. 153
—— —— see M. Kimura. 159
Hayami, R.,—On the equilibrium in the System of Copper Sulphate, Ammonium Sulphate, Potassium Sulphate, and Water at 25°C. 359
Hirata, H.,—see U. Yoshida. 89
Hirose, M.—see T. Sakao. 349
Iitsuka, D.,—Metallographische Untersuchung ueber das System von Antimonsulfid und Bleisulfid. 1
Inamura, K.,—Hydrates of Aluminium Nitrate. 105
Inoue, S.,—Metallographische Untersuchung ueber das System von Wolfram und Blei. 43
Ishino, M.,—On the Negatively Charged Particles in the Positive Ray tube. 355
—— On the Evidance of the Existence of Isotopes of chlorine. 311
Kashima, K.,—see M. Kuhara. 69
Kashiwagi, H.,—see T. Nishiuchi. 273
Kimura, M.,—The spectrum of Bromine I. Line and Band Spectra, Lines of Arc and Spark Types and the Relations between the Lines. 127
—— The Spectrum of Bromine II. Line Structures and the Zeeman Effect. 139
—— In the Distribution of Charged Ions in the Path of an Electric Discharge through a Tube containing Bromine Vapour. 151
—— Studies of the Spectrum of Chlorine I. Emission and Absorption Spectra. ... 155
—— Studies of the Spectrum of Chlorine II. The Influence of Magnetic Field upon Spectral Lines of Chlorine. 159
—— Unilateral Conductivity of Tubes having a Salt Electrode. 163
—— A Short Note on the Arc and Spark Lines of Iodine. 167
—— Figures produced by Crystallization of potassium-bichromate. 173
Konno, K.,—Metallographische Untersuchung ueber das System von Antimonsulfid und Silbersulfid. 51
Kudara, K.,—see I. Yamamoto. 23
Kuhara, M.,—On the Beckmann Rearrangement. IX. 69
—— Constitution of the Salts ot Aminoacetophenones. 201
Makino, K.,—Electrical States of Iodine Vapour when emitting Line and Band Spectra. 169
Mashimo, T.,—A method of investigating the action of ultravialet rays on bacteria. 1
Matsumiya, K.,—On the organic Compounds of Arsenic. I. 217

PAGE.

Matsumoto, T.,—On the Solutions of Partial Differential Equations of the First Order at the singular Points II. 77

Matsumoto T.,—On the Solutions of Partial Differential Equations of the First Order at the singular Points III. 225

—— Some properties of Analytic Elements on and outside Their Circles of Convergence. 393

Matsui, M.,—Electrolytic Reduction of Monthone. 245

—— The Qualitative and Quantative Determination of Nickel and Cabalt... 265

Nagahata, J.,—see M. Kimura. 663

Nakamura, G.,—Luminosity produced by Electric Discharges in Tubes containing Gasses at Slightly Reduced Preasures. 383

Nakazawa, T.,—see M. Matsui. 265

Nishiuchi, T.,—Oriented Circles in Non-Enclidean Space. 273

Osaka, Y.,—Notes on the Normalities of Standard Solutions. 113

—— Potassium Bioxalate and the Standardisation of Alkali Solutions... 371

Saito, H.,—see M. Kuhara. 201

Sakao, T.,—On the Colour of coloured Fluorites. 349

Shimomura, H.,—see M. Kuhara. 201

Shimizu, S.,—see M. Matsui. 245

Takahashi, Y.,—Metallographische Untersuphung ueber das System von Wismutsulfid und Antimonsulfid. 47

Takegami, S.,—A study of the Reciprocal Salt Pairs:
$Na_2Cl_2+MgSo_2 \rightleftarrows Na_3So_4+MgCl_4$. 317

Toda, S.,—Equilibrium in the System: Potassium Chlorate, Potassium Nitrate and Water at 25°C. 377

—— The Oxychlorides of Mercury. Equilibrium in the System: Mercuric Chloride, Yellow Mercuric Oxide and Water at 35°C. 305

Tomoshige, N.,—Metallographishe Untersuchung ueber das System von Wismut und Selen. 55

Ueta, Y.,—see I. Yamamoto. 23

Yamamoto, I.,—Light-curves of Several recent Novae, and some notes on the general features thereof. .... — 13

—— Observations of Nova Aquilae No. 3. 23

Yasui, T.,—Ueber das mit Chrom zusammenfaellende Zink. 65

Yoshida, U.,—Brush and Glow Discharges of Electricity and the Formation of Spark. 89

—— On the Regularsity in the Stark Effect on the Spectral Lines of Hydrogen and Helium. 187

—— On a Mica Ray Spectrometer. 343

INDEX OF SUBJECTS.

PAGE.

Alkali Solutions, potassium bioxalate and the standardisation of (*Osaka and Ando*) ... 371
Aluminium nitrate, hydrates of (*Inamura*). 105
Aminoacetophenones, constraction of the salt of (*Kuhara, Saito and Shimomura*). ... 201
Ammonium sulphate, copper sulphate, potassium sulphate and water at 25°C, on the equilibrium in the system of (*Hayami*). 359
Analytic element, some properties of, on and outside their circles of convergence (*Matsumoto*). 393
Antimonsulfid und Silbersulfid, metallographische Untersuchung ueber das System von (*Konno*). 51
Antimonsulfid und Wismutsulfid, metallographische Untersuchung ueber das System von (*Takahashi*). 47
Antimonsulfid und Bleisulfid, metallographische Untersuchung ueber das System von (*Iitsuka*). 61
Aquilae No. 3, observations of (*Yamamoto, Ueta and Kudara*). 23
Arsenic 1, on organic compounds of (*Matsumiya*). 217
Beckman Rearrangement IX, on the (*Kuhara and Kashima*). 69
Blei und Wolfram, metallographische Untersuchung ueber das System von (*Inouye*). ... 43
Bleisulfid und Antimonsulfid, metallographische Untersuchung ueber das System von (*Iitsuka*). 61
Bromine I., the spectrum of, line and band spectra, lines of arc and spark types and the relations between the lines (*Kimura*). 127
Bromine II., the spectrum of, line structures and Zeeman effect (*Kimura*). 139
Bruch and glow discharges of electricity and the formation of spark (*Yoshida and Hirata*). 89
Charged ions, on the distribution of, in the path of an electric discharge through a tube containing bromine vapour (*Kimura*). 151
Charged particles, on the negatively, in the positive ray tube (*Ishino and Arakatsu*). 355
Chlorine I., studies of the spectrum of, emission and absorption spectra (*Kimura and Fukuda*). 155
Chlorine II., studies of the spectrum of, the influence of magnetic field upon spectral lines of chlorine (*Kimura and Fukuda*). 159
Chlorine, on the evidence of the existence of isotropes (*Ishino*). 311
Colloidal particles, on the limiting size of, in a browian motion (*Arakatsu and Fukuda*). 179
Coloured fluorites, on the colour of (*Sakao and Hirose*). 349
Cobalt and nickel, the qualitative and quantitative determination of (*Matsui and Nakazawa*). 265
Copper sulphate, ammonium sulphate, potassium sulphate and water at 25°C, on the equilibrium in the system of (*Hayami*). 359

PAGE.

Glow and bruch discharges of electricity and the formation of spark (*Yoshida and Hirata*). ... 89
Light-curves of several recent novae, and some notes on the general features thereof (*Yamamoto*). ... 13
Luminosity produced by electric discharges in tubes containing gases at slightly deduced presures (*Nakamura*). ... 383
Iodine vapour, electrical states of, when emitting line and band spectra (*Makino*) ... 169
Iodine, a short note on the arc and spark lines of (*Kimura*). ... 167
Mercuric Chloride, Yellow Mercuric Oxide and Water at 35°C, equilibrium in the system (*Toda*). ... 305
Mercury, the oxychlorides of (*Toda*). ... 305
Metallographische Untersuchung ueber das System von Wolfram und Blei (*Inouye*). ... 43
Metallographische Untersuchung ueber das System von Wismutsulfid und Antimonsulfid (*Takahashi*). ... 47
Metallographische Untersuchung ueber das System von Antimonsulfid und Silbersulfid (*Konno*). ... 51
Metallographische Untersuchung ueber das System von Wismut und Selen (*Tomoshige*). 55
Metallographische Untersuchung ueber das System von Antimonsulfid und Bleisulfid (*Iitsuka*)... ... 61
Mica X-ray spectrometer, on a (*Yoshida*). ... 343
Millet jelly, optical properties of Japanese (*Fukuda*). ... 183
Monthone, electrolytic reduction of (*Matsui and Shimizu*). ... 245
Nickel and cobalt, the qualitative and quantitative determination of (*Matsui and Nakazawa*). ... 265
Normalities of standard solutions, notes on the (*Osaka*). ... 113
Nora Aquila No. 3, observations (*Yamamoto and Ueta*). ... 23
Oriented circles in non-enclidian space (*Nishiuchi and Kashiwagi*). ... 273
Partial differential equations, on the solutions of, of the first order at the singular points II. (*Matsumoto*). ... 77
Partial differential equations, on the solutions of, of the first order at the singular points III. (*Matsumoto*). ... 225
Potassium chlorate, Potassium nitrate and water at 25°C, equilibrium in the system (*Toda*). ... 377
Potassium nitrate, potassium chlorate and water at 25°C, equilibrium in the system (*Toda*). ... 377
Potassium-bichromate, figures produced by crystallization of (*Kimura*). ... 173
Potassium sulphate, copper sulphate, ammonium sulphate and water at 25°C, on the equilibrium (*Hayami*). ... 359
Potassium Bioxalate and the standardisation of alkali solutions (*Osaka and Ando*). ... 371
Qualitative and Quantitative determination of nickel and cobalt (*Matsui and Nakazawa*). 265
Reciprocal salt pairs: $Na_2Cl_2 + MgSo_4 \rightleftarrows Na_2So_4 + MgCl_2$, a study of the (*Takegami*). 317
Selen und Wismut, metallographische Untersuchung ueber das system von (*Tomoshige*). 55
Silbersulfid und Antimonsulfid, metallographische Untersuchung ueber das System von (*Konno*). ... 51
Stark effect, on the regularity in the, on the spectral lines of hydrogen and helium (*Yoshida*). ... 187
Standard solutions, notes on the nomalities of (*Osaka*). ... 113

PAGE.

Sulphur at various Temperatures, the absorption of light by (*Fukuda*)... 351
Ultraviolet rays on bacteria, a method of investigating the action of (*Mashimo*). ... 1
Unilateral conductivity of tubes having a salt electrode (*Kimura and Nagahata*). ... 163
Wismut und Selen, metallographische Untersuchung ueber das System von (*Tomoshige*). 55
Wismut sulfid und Antimonsulfid, metallographische Untersuchung ueber das System von (*Takahashi*). 47
Wolfram und Blei, metallographische Untersuchung ueber das System von (*Inoue*). ... 43
Yellow mercuric oxide, mercuric chloride and water at 35°C equilibrium in the system (*Toda*). 305
Zink, ueber das mit Chrom zusammenfällende (*Yasui*). 65

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CONTENTS.

Potassium Bioxalate and the Standardisation of Alkali Solutions.
By Yukichi Osaka and Kinji Ando. 371

Equilibrium in the System: Potassium Chlorate, Potassium Nitrate and Water at 25°C.
By Shigeru Toda. 377

Luminosity produced by Electric Discharges in Tubes containing Gases at Slightly Reduced Pressures.
By Gisaburo Nakamura. 383

Some Properties of Analytic Elements on and outside Their Circles of Convergence.
By Toshizô Matsumoto 393